MMAT 5010 Linear Analysis (2023-24): Homework 5
Deadline: 02 Mar 2024

Important Notice:

♣ The answer paper must be submitted before the deadline.
♠ The answer paper MUST BE sent to the CU Blackboard. Please refer to the course web for details.

1. Suppose that the Euclidean space $\mathbb{R}^n$ is endowed with the usual norm, that is, $\|x\|_2 := \sqrt{\sum_{k=1}^n |x_k|^2}$ for $x = (x_1, ..., x_n) \in \mathbb{R}^n$. For each $x \in \mathbb{R}^n$, put $\|x\|_\infty := \max\limits_{1 \le k \le n} |x_k|$.
   Using the definition of equivalent norms, show that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent norms. From this, show that if we let $I : (\mathbb{R}^n, \|\cdot\|_\infty) \to (\mathbb{R}^n, \|\cdot\|_2)$ be the identity map, i.e., $I(x) = x$ for all $x \in \mathbb{R}^n$, then the map I and its inverse map I^{-1} both are continuous.

2. Let $X := \{f : [a, b] \to \mathbb{R} : f \quad \text{is continuous on} \quad \mathbb{R}\}$. For each $f \in X$, let $\|f\|_1 := \int_a^b |f(t)|dt$ and $\|f\|_\infty := \sup\{|f(t)| : t \in [a, b]\}$. Put

$$Tf(x) := \int_a^x f(t)dt$$

   for $x \in [a, b]$. Show that $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_\infty)$ is a bounded linear map of norm 1.

$* * *$ **Happy Year of Dragon** $* * *$